Stochastic Economic Dynamics

Bjarne S. Jensen & Tapio Palokangas (Editors)

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Contents in Brief

Table of Contents

Introduction

Bjarne S. Jensen
University of Southern Denmark and Copenhagen Business School

Tapio Palokangas
University of Helsinki and HECER

A *unity* of *aim* and a *diversity* of *topics* shaped the contents of this volume. Although each chapter can be read independently as a self-contained presentation, the book is more than a collection of the individual contributions. Difficult subjects are interrelated, juxtaposed, and examined for consistency in various disciplinary, theoretical, and empirical contexts. The major unifying theme of this joint work is the coherent and rigorous treatment of *uncertainty* and its implications for describing relevant stochastic processes through *basic* (prototype, core) *models* and *differential equations* of *stochastic dynamics*.

Part I: Developments in Stochastic Dynamics

1. Bernt Øksendal. *Fractional Brownian Motion in Finance.*

Stochastic processes in continuous time are here described by a fractional Brownian motion (Wiener process) in which the *stochastic increments* are not necessarily independent. The increments have a *covariance* function, for which the size of the *Hurst parameter* (H) is critically important. When $H = \frac{1}{2}$, the fractional Brownian motion coincides with the classical Brownian motion. If $H > \frac{1}{2}$, the stochastic increments have a positive *autocorrelation* (motion is persistent). If $H < \frac{1}{2}$, increments have a negative autocorrelation (motion is anti-persistent). This chapter gives a survey of the *theory* of *stochastic* *calculus* (integrals) with fractional Brownian motion and discusses the *applications* of f*ractional stochastic calculus* to financial markets. *Asset prices* are described as *solutions* of stochastic differential equations that are driven by the generalized stochastic processes.

2. Bjarne S. Jensen, Chunyan Wang, and Jon Johnsen. *Moment Evolution of Gaussian and Geometric Wiener Diffusions.*

This chapter analyzes two *basic stochastic models*: The time homogeneous Gaussian and the geometric Wiener diffusion of *two-dimensional vector processes.* Using the theory of stochastic processes and Ito's lemma, the probability distributions of the stochastic state vectors are described by the evolution of their moments (*expectation* vector and *covariance* matrix as functions of time). These moments satisfy certain systems of ordinary (deterministic) differential equations. By solving these ODE, the authors present explicit solutions for the first-order and second-order moment functions. Kolmogorov's forward equation is used to derive the results by alternative methods and to gain information on the probability distributions. The general *closed form* results for these *moment evolutions* - still unavailable - have many applications in models of linear *dynamics* with *uncertainty.*

3. Nishioka Kunio. *Two-Dimensional Linear Dynamic Systems with Small Random Terms.*

Chapter 3 links up with chapter 2 and further studies the *asymptotic behavior* of the *time paths* of two-dimensional linear dynamic systems that are perturbed by small random terms. Economic growth is traditionally treated as a non-random dynamic system. If the system is linear and two-dimensional, it can be classified as one of *five* well-known *types*, according to its *long-run* (asymptotic) behavior. With *uncertainty* involved in economic growth, the asymptotic behavior of a system with random perturbations is important to investigate, for example, when analyzing *steady state* properties of economic growth. If the random perturbations are small, the *asymptotic* behavior (time paths) of the linear *stochastic* system is the *same* as in *non-random* cases, unless the relevant dynamic system is a *circle* or a proper *node.*

4. Masao Nagasawa. *Dynamic Theory of Stochastic Movement of Systems.*

The dynamic theory of stochastic movement of systems contains a general mathematical theory of random motion - consisting of two parts, *stochastic kinematics* and *stochastic mechanics*. The stochastic *kinematics* is analytically described by Kolmogorov's PDE equation, which - with its *drift* coefficient and *diffusion* coefficient - *uniquely* characterizes the *transition probability* distribution when an *initial* distribution is prescribed. The stochastic *mechanics* contains the mechanical *equation* of *motion*, which, in addition to the Kolmogorov equation, includes a *potential* function of *external* forces.

The potential function determines a so-called *induced drift* coefficient. This induced drift coefficient in turn enters a new kinematic (Kolmogorov) equation that fully describes the relevant transition probability density of the observed stochastic process. However, Kolmogorov equations are not easy to solve, except in some simple cases. Therefore, to analyze the stochastic processes, it is often better to use Ito's stochastic differential equations (SDE), and in solving them, we have the powerful tools of *sample path* analysis, in particular, Lévy's formula and Ito's formula.

The dynamic theory of stochastic motion (mechanics) is then applied to *Quantum Mechanics* (Schrödinger's complex "wave equation"). Sample paths in one and two dimensions of simple motions governed by Schrödinger's equation are illustrated. Finally, the methodology is applied to the Schrödinger equation with Coulomb potential to obtain the *sample path* of the *electron* in the *hydrogen* atom. Here the *critical* ("attractive") *radius* of the *solved* radial motions (*sample paths*) agrees with the *classic* Bohr *radius* expression for the "stationary states" of hydrogen. The conceptual existence of sample paths (stochastic trajectories) has been controversial (even denied) in quantum dynamics. Because *sample paths* and stochastic differential equations are the natural *generalization* of *deterministic dynamics* in *Economics*, the chapter has devoted a keen effort of calculation to demonstrate some particular sample paths for states of hydrogen motion that are governed by the universal Schrödinger equation.

Part II: Stochastic Dynamics in Basic Growth Models and Time Delays

5. Bjarne S. Jensen and Martin Richter. *Stochastic One-Sector and Two-Sector Growth Models in Continuous Time.*

This chapter extends the basic *deterministic* one-sector and two-sector growth models to a *stochastic* context in continuous time, using *Wiener processes* for the description of various sources of uncertainty in the growth rate of the labor force, the rate of capital depreciation, and the saving rate. The *drift* and *diffusion* coefficients of the stochastic dynamic systems are *homogeneous* of degree one in the two state variables, labor and capital - which allows a reduction to the one-dimensional stochastic dynamics of the capital-labor ratio. The crucial issue of *absorbing boundaries* for the stochastic growth models is rigorously examined, and simple criteria (sufficient conditions) for *inaccessible* boundaries are established, a subject that the literature has not yet adequately addressed. The *steady state probability* distribution of the capital-labor ratios is derived from Kolmogorov's forward equation. For stochastic one-sector and two-sector growth models, the *sample paths* - of the *transition* to steady states or *persistent* (endogenous) growth - of particular state variables are simulated for many parametric specifications of the CD and CES sector technologies involved. The impacts of technology shocks are similarly demonstrated. All relevant sample paths are exhibited on both shorter and longer time horizons.

6. Zhu Hongliang and Huang Wenzao. *Comparative Dynamics in a Stochastic Growth and Trade Model with a Variable Savings Rate.*

The authors consider neoclassical two-sector *growth models* of a *small country* that is *trading* in both commodities in *stochastic environment* in continuous time, and they use a *saving function* for which the rate of saving depends on the capital-labor ratio and a *policy parameter.* The *global comparative* dynamic properties of the capital accumulation process are studied with respect to changes in the policy parameter. By characterizing the entire time path of the capital accumulation process, the effect of the policy parameter can be determined. The *time path* of the capital-labor ratio satisfies a *monotonicity* property if the saving function changes monotonically with respect to a policy parameter. In addition, the *impact* of the policy parameter on the *steady-state distribution* of the capital-labor ratio is analyzed.

7. Zhu Hongliang and Huang Wenzao. *Inada Conditions and Global Dynamic Analysis of Basic Growth Models with Time Delays.*

In economics, time *delays* are often neglected in *continuous time dynamics*, no doubt due to the difficulty in solving and analyzing such models. Nevertheless, economic development depends not only on the *current state*, but also on *past states* (history), so delay phenomena also influence the *dynamic characteristics* of economic systems. This chapter introduces *time delays in* a particular *neoclassical growth* model. The *global conditions* for steady-state *stability* and/or persistent *oscillation* around a steady state are obtained and analyzed. It is shown that oscillations in growing economies are not rare, but common.

8. Morten Brøns and Bjarne S. Jensen. *Hopf Bifurcation in Growth Models with Time Delays.*

Chapter 8 complements the *global analysis* of particular difference-differential equations in chapter 7 by performing a non-linear *local analysis* of the dynamics of the delay model when the size of *time delay* is close to a *critical* value. For this critical value, a *Hopf bifurcation* (of a fixed-point into a closed orbit in the neighborhood of the equilibrium) occurs, that is, *periodic solutions* ("limit cycles") are created when the steady state solution of the capital-labor ratio loses its local stability. Analytical *criteria* are derived to determine the *stability types* (supercritical or subcritical) of the periodic solutions (limit cycles). Finally, it is shown that the delay model with CES production functions can exhibit dynamics with *solutions* which have been observed in other delay-differential equations: square waves and *chaos* (aperiodic waves/cycles). Simulations illustrate the analytical results and theorems.

Part III: Intertemporal Optimization in Consumption, Finance, and Growth

9. Claus Munk and Carsten Sørensen. *Optimal Consumption and Investment Strategies in Dynamic Stochastic Economies.*

The authors derive optimal consumption and investment *strategies* of an *investor* with a CRRA *utility* of consumption and *terminal wealth* and with access to trade in a *complete*, but otherwise very general, *financial market.* Interest rates, excess expected returns, price volatilities, correlations, and consumer prices may all evolve stochastically over time, even with non-Markovian dynamics. The *risks* that individuals want to hedge are shown, as well as how to finance a desired real consumption process by investing in a market of nominal securities. The general results are extended to the case of a HARA utility and power-linear habit utility. The chapter also discusses how labor income and undiversifiable shocks should be included in the consumer price index. In the special case where real *interest rates* are Gaussian and real *market prices of risk* are deterministic, the chapter shows that CRRA *investors hedge* with a single real bond, with a utility of terminal wealth (real zero-coupon bond maturing at the horizon), and with a utility of intermediate consumption (a bond with continuous coupons proportional to the expected future real consumption rate under the forward martingale measure). The *results* are *illustrated* by two examples: (i) non-Markovian HJM term structure dynamics, (ii) stochastic volatility and excess returns in the stock market.

10. Mogens Steffensen. *Differential Systems in Finance and Life Insurance.*

Financial and life insurance mathematics share a common problem of valuation of *future payment streams.* However, the *valuation principles* - *no arbitrage* and *diversification* - differ because risks differ. In both financial and life insurance mathematics, the valuation problem reduces to calculating *conditional expected values* and the extrema of these values. If the *risk process* is Markovian, expected values can be characterized by solutions to systems of *deterministic differential equations.* Deterministic differential systems also appear in financial and life insurance decision-making. They characterize optimal expected values of future utility and optimal decisions. For both valuation and optimization, we derive some *classical* examples from *finance* and *life insurance* and generalize to situations that are relevant in both fields. We study valuation with participation and early exercise options, applications of the linear regulator, and generalized consumption problems. The *collection* of *results* and *proofs* demonstrate both the *similarities* and the small but *important differences* in the various problems.

11. Paul A. de Hek. *Uncertain Technological Change and Capital Mobility.*

Unpredictable variations in economic productivity may have a positive or negative effect on the *average growth rate* of output. This theoretical ambiguity result is not solely determined by the value of the *elasticity* of *intertemporal substitution*. The growth-uncertainty relationship depends on two factors: whether *returns to scale* in *knowledge creation* are increasing or non-increasing, *and* whether the elasticity of intertemporal *substitution* (of profits) is higher or lower than some *critical* value. Empirical studies concerning these two factors indicate that unpredictable variations in economic productivity have a negative effect on the average long-run growth rate.

12. Nils Chr. Framstad. *Stochastic Control, Non-Depletion of Renewable Resources, and Intertemporal Substitution.*

For a wide class of models concerning the *optimal extraction* of a renewable resource, it is well known that an *expected profit maximizer* with an infinite horizon does not deplete the resource completely if its relative growth rate is strictly greater than the discount rate. This principle is extended to *preferences* that have intertemporal substitution in direct utility rates and that exhibit risk aversion (or risk neutrality) sufficiently close to zero. For a CRRA *utility*, the effect of intertemporal substitution is seen more clearly. The model in this chapter is an *Itô process* driven by *semi-martingales.*

13. Klaus Wälde. *Capital Accumulation in a Growth Model with Creative Destruction.*

Capital accumulation and creative destruction are modeled together with risk-averse households. The novel aspect - *risk-averse households* - allows the use of well-known models not only for analyzing long-run *growth* as in the literature, but also short-run *fluctuations.* The model remains *analytically tractable* because of a very convenient property of household investment decisions in this stochastic setup.

14. Tapio Palokangas. *Employment Cycles in a Growth Model with Creative Destruction.*

This chapter constructs a model that would explain economic *growth* with *fluctuations in output and employment.* The particular features of the model are the following. There is *creative destruction* in the sense that a new technology renders an old technology obsolete. There are *efficiency wages* in R&D. In production, there is *union-employer bargaining* over wages. The firms can increase the probability of a technological change for themselves by R&D. *Learning-by-investment* increases the productivity of labor in the consumption-good sector in proportion to the expected accumulation of capital.

The main results are: In the long run, the economy follows a balanced-growth path that satisfies *Kaldor's stylized facts.* Wages in production grow on average in proportion to the level of productivity. The economy generates cycles around this long-run equilibrium. Capital stock swings up and down due to endogenous technological shocks. Because union-firm bargaining keeps real wages in proportion to the level of productivity, the labor-capital ratio is fixed and employment swings in proportion to capital stock. Thus, a stationary state equilibrium is characterized by involuntary unemployment, *employment cycles* and *stable real wages* in production.

Part I: Developments in Stochastic Dynamics

Chapter 1

Fractional Brownian Motion in Finance

Bernt Øksendal
Center of Mathematics for Applications (CMA)
Department of Mathematics, University of Oslo, and
Norwegian School of Economics and Business Administration

1.1 Introduction

How can we model (as a function of time)

(i) the levels of a river?

(ii) the characters of solar activity?

(iii) the widths of consecutive annual rings of a tree?

(iv) the outdoor temperature at a given point?

(v) the values of the *log returns* h_n, defined by

$$h_n = \log \frac{S(t_n)}{S(t_{n-1})}$$

where $S(t)$ is the observed price at time t of a given stock?

And how can we model

(vi) the turbulence in an incompressible fluid flow?

(vii) the electricity price in a liberated electricity market?